REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE

3. REPORT TYPE AND DATES COVERED
Reprint

4. TITLE AND SUBTITLE
TITLE ON REPRINT

5. FUNDING NUMBERS
DAAG55-98-1-0418

6. AUTHOR(S)
AUTHOR(S) ON REPRINT

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
University of Southern California

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
ARO 38076.14-MA

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12 b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

ABSTRACT ON REPRINT

20030515 121

14. SUBJECT TERMS

15. NUMBER IF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18
298-102

J.L. Menaldi et al. (Eds.)
IOS Press, 2001

Stochastic Navier-Stokes Equations. Propagation of Chaos and Statistical Moments

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Abstract

In the first part of the paper, we discuss existence and uniqueness results for a general stochastic Navier-Stokes equation (SNS) derived from the first principles. In the second part, we deal with the propagation of Wiener chaos by the SNS and its relation to statistical moments of the solution.

Key words: Stochastic Navier-Stokes, turbulence, Wiener chaos, moments.

1 Introduction

Relation of the Navier-Stokes equation to the phenomenon of turbulence have fascinated physicists and mathematicians for a long time. One of the popular hypothesis relates the onset of turbulence to the "randomness of background movement". Bensoussan and Temam [1] have pioneered an analytic version of this approach based on investigation of a stochastic Navier-Stokes equation driven by white noise type random force. Later this approach was substantially developed and extended by many authors (see, e.g. [2], [3], [4], [5], [6], [7], [8], [9], [14] etc.).

In these papers, some form of stochastic Navier-Stokes (SNS) equation was postulated at the inception point. A somewhat different, approach was taken in the recent paper [11]. In this papers it was postulated that the dynamics of the fluid particle was given by the stochastic diffeomorphism

† This work was partially supported by NSF Grant DMS-98-02423.

‡This work was partially supported by NSF Grant DMS-98-02423, ONR Grant N00014-97-1-0229, and ARO Grant DAAG55-98-1-0418.

$$\dot{\eta}(t,x)=\mathbf{u}(t,\eta(t,x))+\sigma(t,\eta(t,x))\circ\dot{W},\quad \eta(0,x)=x \tag{1.1}$$

with undetermined (random) local characteristics $\mathbf{u}(t,x)$ and $\sigma(t,x)$. In this setting, $\sigma(t,x)\circ\dot{W}$ models the turbulent part of the velocity field while $\mathbf{u}(t,x)$ models its regular component. Following the classical scheme of the Newtonian fluid mechanics (i.e. coupling (1.1) with Newton's second law), a very general SNS equation (see (2.3) below), was derived. It includes as special cases the classical deterministic Navier-Stokes and Euler equation as well as most of the variations of the SNS equation considered in the literature.

In the present paper we will discus new results on existence and uniqueness of local and global (pathwise) solutions to these equations in the Bessel classes H_p^n, In addition, we will derive a deterministic parabolic system for the Hermite-Fourier coefficients in Wiener chaos expansion of $\mathbf{u}(t,x)$ which we refer to as "propagator". It will be shown that the statistical moments of the velocity field $\mathbf{u}(t,x)$ can be expressed straightforwardly via the solution of the propagator. While still an infinite-dimensional system, the propagator for the SNS equation is a much more simple object then the related Kolmogorov equation. On the other hand, it is quite sufficient for dealing with basic statistical properties of solutions to the SNS equation.

2 Stochastic Navier-Stokes and Euler Equations

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space and Y be a separable Hilbert space. The scalar product in Y will be denoted by $x\cdot y$. Let W be an Y-valued cylindrical Brownian motion on $(\Omega,\mathcal{F},\mathbf{P})$. The $\mathbf{P}$-completion of the $\sigma-$algebra $\bigcap_{\varepsilon>0}\sigma(W(s),s\leq t+\varepsilon)$ will be denoted $\mathcal{F}_t^W$.

Let us assume that the stochastic fluid flow is given by equation (1.1) where $\sigma^i(t,x)$ and $u^i(t,x)$ are, respectively, an $Y-$valued and a real-valued $\mathcal{F}_t^W-$adapted, functions on $\Omega\times[0,\infty)\times R^d$. Throughout what follows the symbol $\circ dW$ (respectively, $\cdot dW$) indicates that the integral is understood in the Stratonovich (respectively, Ito) sense. F stochastic integrals are understood in the sense of [9], [10]. (The Kunita and Walsh integrals as well as the integrals with respect to spatially-homogeneous Brownian motion are included as special cases of the above setting).

In contrast to the classical assumption that the velocity field $\mathbf{u}(t,x)$ is differentiable in t, we will suppose that it is a semimartingale. More specifically, we will assume the following:

H1. $d\mathbf{u}(t,x) = \alpha(t,x)\,dt + \beta(t,x)\cdot dW(t)$ where α and β are $\mathcal{F}_t^W$ – adapted vector-functions on $\Omega \times [0,\infty) \times R^d$ taking values in R^d and Y^d, respectively.

Note that as in the classical theory, the components of the velocity field, α, β, σ, are not supposed to be given, and will be determined later from to the conservation of momentum principle. The only additional assumption we make regarding these functions is that they are sufficiently smooth in x and the related integrals are defined.

By the Newton 2nd law, we have $\ddot{\eta}(t) = \mathbf{F}(t,\eta(t))$ where $\mathbf{F}(t,x)$ is the total force applied to the fluid particle.

Since the acceleration and the force must have the same structure, we conclude (for more detail, see [11]) that there exist $\mathcal{F}_t^W$-adapted functions $\mathbf{F}_a$, $\mathbf{F}_\partial$ and $\mathbf{F}_t$, so that

$$\begin{aligned} &\int \varphi(t)\,\mathbf{F}(t,x)\,dt = \int \varphi(t)\,(\mathbf{F}_a(t,x)\,dt + \mathbf{F}_\partial(t,x)\cdot dW(t)) \\ &-\int \varphi'(t)\,(\mathbf{F}_t(t,x)\cdot dW(t) + (1/2)\,\partial_i \mathbf{F}_t(t,x)\cdot F_t^i(t,x))\,dt. \end{aligned} \tag{2.2}$$

Similarly to the classical setting, we assume that the forces acting on the fluid particle include pressure and body forces. More specifically, we assume that $\mathbf{F}_a(t,x) = -\nabla p_a(t,x) + \mathrm{f}_a(t,x)$, $\mathbf{G}(t,x) = -\nabla\tilde{p}(t,x) + \mathrm{g}(t,x)$, and $\mathbf{D}(t,x) = -\nabla p_t(t,x) + \mathrm{f}_t(t,x)$. The components of the body force are considered to be given while the components of the pressure are subject to determination.

It was shown in [11] that (1.1), (2.2), and Newton's 2nd law yield the following equations for the components of the velocity field :

$$\begin{cases} \partial_t \mathbf{u} = \partial_j(a^{ij}\partial^i \mathbf{u}) - u^j\partial_j \mathbf{u} - (\nabla\sigma^i)\cdot\partial_i\tilde{p} + \mathrm{f} - \nabla p + \\ (\mathrm{g} - \nabla\tilde{p} - \sigma^i\partial_i\mathbf{u})\cdot\dot{W}, div\,\mathbf{u} = 0, \mathbf{u}(0,\mathrm{x}) = \mathbf{u}_0(\mathrm{x}) \end{cases} \tag{2.3}$$

and

$$\sigma(t,x) = -\nabla\tilde{p}(t,x) + \mathbf{d}(t,x)$$

where $p = p_a - \sigma^i\cdot\partial_i\tilde{p}$, $\mathrm{f} = \mathrm{f}_a - \sigma^i\cdot\partial_i\mathrm{g}$, and $a^{ij} = \frac{1}{2}\sigma^i\cdot\sigma^j$ in the case of ideal fluid and $a^{ij} = \nu\delta_{ij} + \frac{1}{2}\sigma^i\cdot\sigma^j$ if the fluid is viscous with the viscosity constant ν.

3 Existence and Uniqueness of Solutions

In this section we will discuss the solvability problem for the equation (2.3) in the Bessel classes. The following notation will be used in the future:

Let $p \in [2, \infty)$ and $n \in (-\infty, \infty)$.

$H_p^n = H_p^n(\mathbf{R}^d)$ is the space of generalized functions u so that $|u|_{n,p} = |(1-\Delta)^{n/2}u|_p < \infty$, where $|\cdot|_p$ is the L_p-norm;

$\mathbb{H}_p^n = \mathbb{H}_p^n(\mathbf{R}^d)$ is the space of all vector fields $\mathbf{u} = (u^1, \ldots, u^d)$ such that $|\mathbf{u}|_{n,p} = (\sum_l |u^l|_{n,p}^p)^{1/p} < \infty$.

We will omit the sub- and super-indices n if $n = 0$.

$\mathbb{L}_p(Y)$ is the space of vector functions with $Y-$valued components g^l such that $|g|_p = (\sum_l |g^l|_p^p)^{1/p} < \infty$;

$\mathbb{L}_p = \mathbb{H}_p$,

$\mathcal{S}_0^\infty(\mathbf{R}^d)$ is the space of all vector fields $\phi = (\phi^1, \ldots, \phi^d)$ such that $\phi^i \in C_0^\infty(\mathbf{R}^d)$ and *div* $\phi = 0$;

The scalar product in $\mathbb{L}_2$ is denoted $(\cdot,\cdot)_0$.

We will need the following assumptions:

B1. There exist constants $0 < K < \infty$ and $\delta > 0$ so that **P**-a.s.

$$(|\partial a^{ij}(t,x)| + |a^{ij}(t,x)| + \sum_{k=0}^{2} |\partial^k \sigma^i(t,x)|_Y) \leq K,$$

and

$$K|\lambda|^2 \geq [a^{ij}(t,x) - \frac{1}{2}\sigma^i(t,x)\cdot\sigma^j(t,x)]\lambda^i\lambda^j \geq \delta|\lambda|^2$$

for all $t \geq 0$ and $x, \lambda \in \mathbf{R}^d$.

B2(p). For each $T > 0$ there are constants C_1 and C_2 such that **P**-a.s. for each $s \leq T$ and $x \in \mathbf{R}^d$,

$$\sum_{k=0}^{2} |\partial_x^k g(s,x)|_Y \leq C_1(s,x) \text{ and } \sum_{k=0}^{2} |\partial_x^k \mathbf{f}(s,x)| \leq C_2(s,x)$$

and $|C_1(t)|_p^p + |C_2(t)|_p^p \leq K$.

The following two results establish strong (pathwise) existence and uniqueness of local and global maximal solutions for the stochastic Navier-Stokes equation (2.3).

Theorem 3.1 *a) Let $p > d$. Assume B1, B2(p) , and suppose that* $|\mathbf{u}$
∞ $\mathbf{P}$*-a.s. Then there is a unique predictable sopping time* ζ, $\mathbf{P}(\zeta$:
1, *such that for each stopping time $S, [0, S] \subseteq [0, \zeta)$ if and only if ther*
$\mathbb{H}_p^1$*-valued continuous solution to (2.3) on $[0, S]$; Also, there is an* $\mathbb{H}_p^1$-
continuous process $\mathbf{u}(t)$ *on* $[0, \zeta)$ *such that* $\mathbf{u}(\zeta-)|_{1,p} = \infty$ *on* $\{\zeta < \infty$
for each S such that $[0, S] \subseteq [0, \zeta)$, $\mathbf{u}(t \wedge S)$ *is a unique solution to (*
$[0, S]$. *Moreover, for* η=*curl* $\mathbf{u}$, $\mathbf{P}$*-a.s. we have*

$$\int_0^S |\nabla \eta(s)|_p^p \, ds < \infty.$$

The function $\mathbf{u}(t)$ and the stopping time ζ are usually referred t
maximal solution and its explosion time.

The proof of the Theorem is quite involved and could not be pre
in this paper. The interested reader could find a complete proof
forthcoming paper [12].

Now let us specialize to the $2D$ case.

Theorem 3.2 *Suppose that the assumptions of the Theorem 3.1 ho*
addition, let us assume that $d = 2$, B2(2) holds, and $\mathbf{P}$*-a.s.* $|\mathbf{u}_0|_{1,p} + |\mathbf{u}$
∞ $\mathbf{P}$*-a.s. Then there exists a unique continuous* $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$*-valued solutic*
to (2.3) on $[0, \infty)$. *Moreover, for each* $T > 0$

$$\mathbf{E} \sup_{s \le T} (|\mathbf{u}(t)|_{1,p}^p + |\mathbf{u}(t)|_{1,2}^p) < \infty$$

provided that

$$\mathbf{E}(|\mathbf{u}_0|_{1,p}^p + |\mathbf{u}_0|_{1,2}^p) < \infty.$$

Proof. The existence of a maximal solution follows from Theore
It remains to prove that $\mathbf{P}(\zeta = \infty) = 1$. Let $y_t = |\mathbf{u}(t)|_{1,p}^p + |\mathbf{u}$
$R_m = \inf(t : y_t \ge m) \wedge \zeta$. Since the sequence R_m "announces" the predi
stopping time ζ, for each $T > 0$

$$\mathbf{P}(R_m < T) \le \mathbf{P}(y_{Rm} \wedge T \ge m) \le m^{-1} \mathbf{E} y_{R_m \wedge T},$$

It can be shown (see [12]) that

$$\sup_m \mathbf{E} y_{R_m \wedge T} < \infty.$$

So $\lim_m \mathbf{P}(R_m < T) = 0$, and therefore $\mathbf{P}(\zeta = \infty) = 1$. For the moments estimates see [12].

Remark. One can consider a more general equation

$$\begin{cases} \partial_t u^l(t,x) = \partial_i \left(a^{ij}(t,x)\, \partial_j u^l(t,x)\right) - u^k(t,x)\, \partial_k u^l(t,x) \\ -\partial_l p(t,x) + b^i(t,x)\partial_i u(t,x) + \partial_i \tilde{p}(t,x) h^{l,i}(t,x) \\ +f^l(t,x,\mathbf{u}(t,x), \nabla \mathbf{u}(t,x)) + \\ +[\sigma^i(t,x)\partial_i u^l(t,x) + g^l(t,x,\mathbf{u}(t,x)) - \partial_l \tilde{p}(t,x)]\, \dot{W}_t, \\ \operatorname{div} \mathbf{u} = 0, \\ \mathbf{u}(0,x) = \mathbf{u}_0(x), \quad l = 1, \dots d, \end{cases} \tag{3.4}$$

where the free forces are functionals of the solution. The results similar to the above hold for equation (3.4) .Of course in this case additional assumptions on regularity of f^l and g^l with respect to $\mathbf{u}, \nabla\mathbf{u}$ as well as the appropriate growth conditions must be introduced (see [12].)

4 Wiener Chaos Expansion and Statistical Moments

In this Section we investigate how the SNS equation (2.3) propagates chaos generated by the driving Brownian motion. Then we will apply the Wiener Chaos expansion to derive formulas for the statistical moments of the velocity field $\mathbf{u}$. For the sake of simplicity, everywhere in this section it will be assumed that the σ, $\mathbf{f}, g$, and $\mathbf{u}_0$ are nonrandom and the assumptions of Theorem 3.2 are in force.

To begin with, we shall introduce the Wiener chaos generated by W.

Let us fix a positive number $T < \infty$. Let $\{m_k, k \leq 1\}$ be an orthonormal basis in $L_2(0,T)$ and $\{\ell_k, k \geq 1\}$ an orthonormal basis in Y. Write $M_i^k(t) = \int_0^t m_i(s)\, dw^k(t)$ where $w^k(t) = (W(t), \ell_k)_Y$. Below in this section we will also use the notation $\xi_i^k = M_i^k(T)$. Let $\alpha = \left\{\alpha_i^k,\ k = 0,1,2,...; i = 1,2,...\right\}$, be a multiindex, i.e. for every (i,k), $\alpha_i^k \in I\!N = \{0,1,2,\dots\}$. We shall consider only such α that $|\alpha| = \sum_{k,i} \alpha_i^k < \infty$, i.e., only a finite number of α_i^k

is non- zero, and we denote by $\mathcal{J}$ the set of all such multiindice write $\zeta_\alpha := \prod_{i=1}^{\infty}\prod_{k=0}^{\infty} H_{\alpha_i^k}(\xi_i^k)$ where H_n is the n^{th} Hermit The random variable $\xi_\alpha = \zeta_\alpha/\sqrt{a!}$ is often referred to as α^{th} (polynomial).

Let $\{\mathrm{e}_i, i \in \mathbb{N}\}$ be an orthonormal basis in $\mathbb{L}_2$. Since, by t Martin Theorem (*see, e.g.*[13]), $\{\xi_\alpha, \alpha \in \mathcal{J}\}$ is an orthonormal $L_2\left(\Omega, \mathcal{F}_T^W, \mathrm{P}\right)$, we have that $\{\mathrm{e}_i \bigotimes \xi_\alpha, i \in \mathbb{N}, \alpha \in \mathcal{J}\}$ is an orth in $L_2\left(\Omega, \mathcal{F}_T, \mathrm{P}; \mathbb{L}_2\right)$. This implies in particular that for every $\mathbf{v} \in$ we have the following Wiener chaos expansion :

$$\mathbf{v} = \sum_{\alpha \in \mathcal{J}} \hat{\mathbf{v}}_\alpha \zeta_\alpha$$

where $\hat{\mathbf{v}}_\alpha = \frac{1}{\alpha!}\mathbf{E}[\mathbf{v}\zeta_\alpha] = \frac{1}{\alpha!}\sum_{i=1}^{\infty}\mathbf{E}[(\mathbf{v}, \mathrm{e}_i)\zeta_\alpha]\mathrm{e}_i$. We will refer unnormalized Hermite-Fourier coefficient of $\mathbf{v}$ (with respect $\{\mathrm{e}_i \bigotimes \xi_\alpha, i \in \mathbb{N}, \alpha \in \mathcal{J}\}$) or simply, Hermite-Fourier coefficient.

By Theorem 3.2, $\mathbf{E}\sup_{s\leq T}|\mathbf{u}(t)|_{1,2}^2 < \infty$. Thus the solution o the Wiener Chaos expansion $\mathbf{u}(t,x) = \sum_{\alpha\in\mathcal{J}} \hat{\mathbf{u}}_\alpha(t,x)\zeta_\alpha$. Of cou problem of interest is how to characterize the Hermite-Fourie $\hat{\mathbf{u}}_\alpha(t,x)$. It will be shown below that these coefficients verify a linear parabolic system of equations. To formulate this stateme cisely, we need some more notation.

Write

$$(\hat{\mathbf{u}}(t) \star \partial_i \hat{\mathbf{u}})_\alpha =$$

$$-\sum_{p\in\mathcal{J}}\sum_{0\leq\beta\leq\alpha}\frac{1}{\alpha!}\binom{p+\beta}{p}\binom{p+\alpha-\beta}{p}p!\hat{u}_{p+\alpha-\beta}^i(t)\,\partial_i\hat{\mathbf{u}}_{p+\beta}(t)$$

$$\mathcal{M}\hat{\mathbf{u}}_\alpha(t) = -\sigma^j(t)\partial_j\hat{\mathbf{u}}_\alpha(t) + I_{\{\alpha=0\}}g(t),$$

For $\mathbf{v} \in \mathbb{L}_2(Y)$, write $\mathcal{G}^j(\mathbf{v}) = \partial_j \int \Gamma_{x_i}(x-y)v^i(y)\,dy$ where $\Gamma(x -$ $y|/2\pi$. The operator $\mathcal{G} = \{\mathcal{G}^j\}_{1\leq i\leq d}$ is often referred to as gradie on $\mathbb{L}_2(Y)$. It is well known that

$$\mathbb{L}_2(Y) = \mathcal{G}(\mathbb{L}_2(Y)) \oplus \mathcal{S}(\mathbb{L}_2(Y))$$

where $\mathcal{S}(\mathbb{L}_2(Y)) = \{g \in \mathbb{L}_2(Y) : \mathrm{div}\, g = 0\}$.

For $\alpha \in \mathcal{J}$, we define multiindex $\alpha(i,j) \in \mathcal{J}$ by the formula

$$\alpha(i,j)_l^k = \begin{cases} \alpha_l^k \text{ if } (k,l) \neq (i,j) \\ \left(\alpha_l^k - 1\right) \wedge 0 \text{ if } (k,l) = (i,j), \end{cases} \tag{4.7}$$

i.e. the multiindex $\alpha(i,j)$ might differ from α only by its (i,j) entry which is equal to $\left(\alpha_j^i - 1\right) \wedge 0$. Finally, write

$$D\mathcal{M}\left(\hat{\mathbf{u}}_\alpha(t)\right) = \left(g^k \alpha_i^k I_{\{|\alpha|=1\}} - \sigma^j(t)\partial_j \hat{\mathbf{u}}_{\alpha(i,k)}(t)\right) m_i(t).$$

Theorem 4.1 *Let* $\mathbf{u}$ *be the maximal solution of the stochastic Navier-Stokes equation* (2.3). *Then the Fourier-Hermite coefficients* $\hat{\mathbf{u}}_\alpha$ *are continuous* $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$*-valued functions on* $[0,\infty)$ *and for each* $T > 0$,

$$\sup_{s \leq T} (|\hat{\mathbf{u}}_\alpha(t)|_{1,p}^p + |\hat{\mathbf{u}}_\alpha(t)|_{1,2}^p) < \infty$$

Moreover, $\{\hat{\mathbf{u}}_\alpha(t,x), \alpha \in \mathcal{J}\}$ *is the unique solution of the system*

$$\begin{cases} (\hat{\mathbf{u}}_\alpha(t), \phi)_0 = I_{\{\alpha=0\}}(\mathbf{u}_0, \phi)_0 + \int_0^t \{-(a^{ij}\partial^i \hat{\mathbf{u}}_\alpha(s), \partial_j \phi)_0 - \\ ((\hat{\mathbf{u}}(s) \star \partial_i \hat{\mathbf{u}}(s))_\alpha, \phi)_0 + (\nabla \sigma^i(t)) \cdot \mathcal{G}^i(\mathcal{M}(\mathbf{u}_\alpha(t)), \phi)_0 + \\ I_{\{\alpha=0\}}(\mathbf{f}(s), \phi)_0 + (D\mathcal{M}(\hat{\mathbf{u}}_\alpha(s)), \phi)_0\} ds, \ div\ \hat{\mathbf{u}}_\alpha = 0, \\ \text{for all } \phi \in \mathcal{S}_0^\infty(\mathbf{R}^d) \text{ and } \alpha \in \mathcal{J}. \end{cases} \tag{4.8}$$

Sketch of Proof. The first part of the statement follows from Theorem 3.2. It can be shown that the relation $\mathbf{E}(\partial_i \mathbf{u}(t) u^i(t) \zeta_\alpha) = (\hat{\mathbf{u}}(t) \star \partial_i \hat{\mathbf{u}})_\alpha$ follows from the well known formula

$$\zeta_\alpha \zeta_\beta = \sum_{p \leq \alpha \wedge \beta} \binom{\alpha}{p} \binom{\beta}{p} p! \zeta_{\alpha+\beta-2p}.$$

Write $\zeta_\alpha(t) = \mathbf{E}[\zeta_\alpha | \mathcal{F}_t]$. Note that $\zeta_\alpha(t)$ verifies the equation

$$d\zeta_\alpha(t) = m_i(t) \alpha_i^k \zeta_{\alpha(i,k)}(t) dw^k(t). \tag{4.9}$$

Now, the equation (4.8) can be derived by differentiating the product $\mathbf{u}(t,x)\zeta_\alpha(t)$ by Ito formula and taking expectations of both sides of the resulting equation.

Making use of the Wiener chaos expansion (4.5)for a solution of the SNS (2.3), one can immediately compute the first two moments of the solution via the Hermite-Fourier coefficients provided by the equation (4.8) for the propagator. Indeed, since $\mathbf{E}\zeta_\alpha = 0$ for $\alpha \neq 0$ and $\mathbf{E}\zeta_0 = 1$ where 0 is the zero length element of $\mathcal{J}$, we have

$$\mathbf{E}\mathbf{u}(t,\mathbf{x}) = \hat{\mathbf{u}}_0(t,\mathbf{x}).$$

By (4.5) and Parceval's identity, one has that for almost all $\mathbf{x},\mathbf{y} \in \mathbf{R}^d$ and $t,s \in [0,T]$,

$$\mathbf{E}\mathbf{u}(t,\mathbf{x})\,\mathbf{u}(s,\mathbf{y}) = \sum_{\alpha\in\mathcal{J}} \hat{\mathbf{u}}_\alpha(t,\mathbf{x})\,\hat{\mathbf{u}}_\alpha(s,\mathbf{y}).$$

Moreover, we have

$$\mathbf{E}\,|\mathbf{u}(t)|^2_{\mathbf{L}_2} = \sum_{\alpha\in\mathcal{J}} |\hat{\mathbf{u}}_\alpha(t)|^2_{\mathbf{L}_2}.$$

Similarly, given the solution of the equation (4.8), the higher order moments of the solution to SNS equation (2.3)can be obtained by computing the moments of the Wick polynomials ζ_α. For example,

$$\mathbf{E}u^i(t,x)\,u^j(t,y)\,u^k(t,z) =$$

$$\sum_{\alpha,\beta,\gamma\in\mathcal{J}'} \frac{\hat{u}^i_\alpha(t,x)\,\hat{u}^j_\beta(t,y)\,\hat{u}^k_\gamma(t,z)}{((\alpha+\beta-\gamma)/2)!\,((\alpha-\beta+\gamma)/2)!\,((\beta-\alpha+\gamma)/2)!}$$

where $\mathcal{J}' = \{\alpha,\beta,\gamma \in \mathcal{J} : \alpha+\beta-\gamma = 2p, p \in \mathcal{J}, 0 \le p \le \alpha \wedge \beta\}$.

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